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Interactions of type IIB D-branes from D-instanton matrix model

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Abstract

We compute long-distance interaction potentials between certain 1/2 and 1/4 supersymmetric D-brane configurations of type IIB theory, demonstrating detailed agreement between classical supergravity and one-loop instanton matrix model results. This confirms the interpretation of D-branes as described by classical matrix model backgrounds as being 'populated' by large number of D-instantons, i.e. as corresponding to non-marginal bound states of branes of lower dimensions. In the process, we establish precise relation between matrix model expressions and non-abelian F^4 terms in the super Yang-Mills effective action.

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1 Introduction

The aim of this paper is to discuss interactions between some D-branes in type IIB matrix model of [1] (see also [2]). Our approach will be that of [3] where the $D = 10$ $U(N)$ super Yang-Mills theory reduced to a point was not related to the (Schild form of) type IIB string action as in [1, 4] but was interpreted as the direct D-instanton counterpart of the D0-brane matrix model of [5]. The two matrix models can be put into correspondence using T-duality in the time direction.

The Dp-brane configurations in the instanton matrix model can be described [1, 3, 6, 7, 8] in a similar way as in the 0-brane matrix model [5, 9, 10]. As was pointed out in [3], they should be identified not with 'pure' type IIB D-branes but with D-branes 'populated' by large number of D-instantons just like D-branes in the matrix model of [5] are 'populated' by large number of 0-branes [11, 12].

In what follows we shall confirm this interpretation by demonstrating that the corresponding long-distance interaction potentials computed in the matrix model and in supergravity are in precise agreement. The matrix model (SYM) result is the same as the short-distance limit of the 1-loop open string theory amplitude while the supergravity result is the long-distance limit of the tree-level closed string theory potential. They agree in the $N \to \infty$ limit in which the brane configurations become supersymmetric for the same reason as in the 0-brane matrix model [5, 11].

The $U(N)$ SYM theory reduced to a point describes a collection of N D-instantons [13, 14]. When some of the ten euclidean dimensions are compactified on a torus T^{p+1}, the classical backgrounds represented by constant abelian fluxes ($[A_m, A_n] = iF_{mn}$) correspond [3] to 1/2 supersymmetric non-marginal bound states of type IIB Dp-branes (i.e. $1 + i$, $3+1+i$, $5+3+1+i$) wrapped over the dual torus $\tilde{T}^{p+1}$. The configuration with self-dual strength $[A_m, A_n]$ represents the 1/4 supersymmetric marginal bound state of D3-brane and D-instantons which we shall denote as $3\|i$ [15, 16].

There is a close T-duality relation to similar configurations in 0-brane matrix model [9, 10, 11, 12]. Indeed, the interaction potentials between such D-branes in the instanton matrix model computed below are direct counterparts of the corresponding results in M(atrix) theory found in [5, 17, 11, 12] for interactions between 1/2 supersymmetric branes and in [18] for interactions involving 1/4 supersymmetric branes.

We shall consider two examples:

(i) interaction between D-instantons and 1/2 supersymmetric 'Dp-branes', i.e. non-marginal $p + (p-2) + \cdots + 1 + i$ bound states;

(ii) interaction between 'D-string', i.e. $1 + i$ bound state, and 1/4 supersymmetric marginal 3-brane–instanton bound state $3\|i$.

In section 2 we shall determine the corresponding closed string theory (supergravity) potentials using classical probe method (see [18] and refs. there). In section 4 we shall

reproduce the same expressions by a one-loop calculation in the instanton matrix model. In section 3 we shall present some general results about 1-loop effective action in $D \leq 10$ SYM theories and explain their relation to the matrix model computations of the leading terms in long-distance interaction potentials.

One natural generalisation of the present work is to 1/8 supersymmetric bound states probed by D-instantons or other type IIB 'D-branes'. In particular, one may consider D-brane configurations corresponding to $D = 5$ black holes as in [19, 20] and [21, 22, 23].

2 Closed string theory (supergravity) description

2.1 D-instanton – 'Dp-brane' interaction

To determine the D-instanton–'Dp-brane' interaction potential we shall consider the latter, i.e. the $p + (p-2) + \cdots + 1 + i$ bound state of type IIB D-branes ($p = -1, 1, 3, 5$) as a probe moving in the classical D-instanton background.[1] This probe can be described, as in [18], by the standard Dp-brane action with a constant world-volume gauge field background. The relevant terms in the euclidean Dp-brane action are ($m, n = 1, ..., p+1$; $i, j = p+2, ..., 10$)

$$I_p = -T_p \Big[\int d^{p+1}x e^{-\phi} \sqrt{\det(G_{mn} + G_{ij}\partial_m X^i \partial_n X^j + \mathcal{F}_{mn})} - \int_{p+1} \sum_k C_{2k} \wedge e^{\mathcal{F}} \Big] , \quad (2.1)$$

where $\mathcal{F}_{mn} \equiv T^{-1} F_{mn}$ (in what follows $B_{mn} = 0$) and C_{2k} is the RR $2k$-form potential. We used the static gauge and took the target-space metric in the block-diagonal form. In general, Dp-brane tension is [24]

$$T_p \equiv n_p \bar{T}_p = n_p g^{-1} (2\pi)^{(1-p)/2} T^{(p+1)/2} , \qquad T \equiv (2\pi\alpha')^{-1} . \quad (2.2)$$

We shall assume that the euclidean world-volume of a type IIB Dp-brane is wrapped over a (rectangular) torus T^{p+1} with volume $V_{p+1} = (2\pi)^{p+1} R_1 ... R_{p+1}$ and that there is a constant world-volume gauge field background

$$\mathcal{F}_{mn} = \begin{pmatrix} 0 & f_1 & & & \\ -f_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & f_l \\ & & & -f_l & 0 \end{pmatrix} , \qquad l \equiv \tfrac{1}{2}(p+1) . \quad (2.3)$$

The Dp-brane with the flux (2.3) on its world-volume represents the non-marginal bound state $(p + (p-2) + \cdots + 1 + i)$ of D-branes of dimensions $p, p-2, ...$ [15] (with branes

[1] By the potential we shall mean the interaction part of the euclidean action. The euclidean time coordinate may be assumed to belong to the internal $p+1$-dimensional torus. Alternatively, one may consider the p-branes discussed below as being '$(p+1)$-instantons' [15], with the time coordinate being orthogonal to the internal torus.

of 'intermediate' dimensions being wrapped over different cycles of the torus). The total numbers of branes of each type are

$$n_{p-2} = n_p 2\pi T \sum_{k=1}^{l} f_k R_{2k-1} R_{2k} \ , \quad ... \quad , \qquad n_{-1} = n_p V_{p+1} \prod_{k=1}^{l} (\frac{T f_k}{2\pi}) \ , \tag{2.4}$$

as can be read off from the Chern-Simons terms in the D-brane action (2.1) [25].

The D-instanton background 'smeared' in the directions of the torus T^{p+1} ($x_1 = y_1, \ ..., \ x_{p+1} = y_{p+1}$) is [26][2]

$$ds_{10}^2 = H_{-1}^{1/2} (dy_1^2 + \cdots + dy_{p+1}^2 + dx_i dx_i) \ , \tag{2.5}$$

$$e^\phi = H_{-1} \ , \qquad C_0 = H_{-1}^{-1} - 1 \ , \qquad H_{-1} = 1 + \frac{Q_{-1}^{(p+1)}}{r^{7-p}} \ , \qquad r^2 = x_i x_i \ .$$

We shall use the notation $Q_p^{(n)}$ for the coefficient in the harmonic function $H_p = 1 + \frac{Q_p^{(n)}}{r^{7-p-n}}$ of p-brane background which is smeared in n transverse toroidal directions. In general,

$$Q_p = N_p g (2\pi)^{(5-p)/2} T^{(p-7)/2} (\omega_{6-p})^{-1} \ , \qquad \omega_{k-1} = 2\pi^{k/2}/\Gamma(k/2) \ , \tag{2.6}$$

$$Q_p^{(n)} = N_p g (2\pi)^{(5-p)/2} T^{(p-7)/2} (V_n \omega_{6-p-n})^{-1} = N_p N_{p+n}^{-1} Q_{p+n} (2\pi)^{n/2} T^{n/2} V_n^{-1}, \tag{2.7}$$

where V_n is the volume of the flat internal torus.

Substituting the background (2.5) into the Dp-brane action (2.1) and ignoring the dependence of X_i on world-volume coordinates x_m (so that the matrix under the square root in (2.1) becomes $H_{-1}^{1/2}\delta_{mn} + \mathcal{F}_{mn}$) we find

$$I_p = -T_p V_{p+1} \Big[H_{-1}^{-1} \prod_{k=1}^{l} \sqrt{H_{-1} + f_k^2} - (H_{-1}^{-1} - 1) \prod_{k=1}^{l} f_k \Big] \ . \tag{2.8}$$

Defining the 'interaction potential' $\mathcal{V}(r)$ ($r^2 = X_i X_i$) as the deviation from the 'free' action of the non-marginal $p + ... + i$ bound state,

$$I_p = I_p^{(0)} - \mathcal{V} = -T_p V_{p+1} \prod_{k=1}^{l} \sqrt{1 + f_k^2} - \mathcal{V} \ , \tag{2.9}$$

we get for the leading long-distance term in $\mathcal{V}$

$$\mathcal{V} = \frac{1}{r^{7-p}} Q_{-1}^{(p+1)} T_p V_{p+1} \ \prod_{m=1}^{l} \sqrt{1 + f_m^2} \ \Big[\sum_{k=1}^{l} \frac{1}{2(1 + f_k^2)} + \prod_{k=1}^{l} \frac{f_k}{\sqrt{1 + f_k^2}} - 1 \Big] + O(\frac{1}{r^{2(7-p)}}) \ . \tag{2.10}$$

The coefficient here is

$$Q_{-1}^{(p+1)} T_p V_{p+1} = 2^{3-l} \ (3 - l)! \ T^{l-4} \ n_p N_{-1} \ , \qquad p = 2l - 1 \ . \tag{2.11}$$

[2] We use the symbol 'i' and subscript '-1' to denote D-instantons and the corresponding quantities.

In the limit of the large background field $\mathcal{F}_{mn}$ ($f_k \gg 1$), i.e. for large instanton 'occupation number' n_{-1} (2.4), we find (we assume that $l \leq 3$ and set $T = 1$)[3]

$$\mathcal{V} = -\frac{1}{r^{8-2l}} 2^{-l}(3-l)!\ n_p N_{-1} \prod_{m=1}^{l} f_m \left[2\sum_{k=1}^{l} f_k^{-4} - (\sum_{k=1}^{l} f_k^{-2})^2 \right] + \dots . \tag{2.12}$$

For example, in the case of $p = 1$, i.e. the D-instanton–'D-string' interaction

$$\mathcal{V} = -\frac{1}{r^6} n_1 N_{-1} \tilde{f}^3 + \dots , \qquad \tilde{f} \equiv f_1^{-1} . \tag{2.13}$$

Note that the potential (2.12) vanishes for $p = 3$ and $f_1 = f_2$. In this case the background field $\mathcal{F}_{mn}$ is self-dual and the interaction between D-instanton and $3+1+i$ non-marginal bound state becomes essentially the same as the interaction between D-instanton and $3\|i$ marginal bound state[4] but '$i - (3\|i)$' is a BPS configuration [15]. Analogous conclusion is reached in the T-dual case of 0-brane $-\ 4+2+0$ bound state interaction: when the magnetic flux on 4-brane is self-dual, $0-(4+2+0)$ interaction is the same as the $0-(4\|0)$ one [18].

The expression (2.12) can be put in the following 'covariant' form

$$\mathcal{V} = -\frac{1}{r^{8-2l}} 2^{-l}(3-l)!\ n_p N_{-1} \sqrt{\det \mathcal{F}_{mn}} \left[\tilde{\mathcal{F}}_{mk}\tilde{\mathcal{F}}_{kn}\tilde{\mathcal{F}}_{ns}\tilde{\mathcal{F}}_{sm} - \frac{1}{4}(\tilde{\mathcal{F}}_{mn}\tilde{\mathcal{F}}_{mn})^2 \right] + \dots , \tag{2.14}$$

$$\tilde{\mathcal{F}}_{mn} \equiv (\mathcal{F}_{nm})^{-1} .$$

Since the D-instanton number in (2.4) is equal to

$$n_{-1} = n_p (2\pi)^{-l} V_{2l} \sqrt{\det \mathcal{F}_{mn}} , \tag{2.15}$$

we can represent (2.14) also as

$$\mathcal{V} = -\frac{(3-l)!\ \tilde{V}_{2l}}{(4\pi)^l r^{8-2l}}\ n_{-1} N_{-1} \left[\tilde{\mathcal{F}}_{mk}\tilde{\mathcal{F}}_{kn}\tilde{\mathcal{F}}_{ns}\tilde{\mathcal{F}}_{sm} - \frac{1}{4}(\tilde{\mathcal{F}}_{mn}\tilde{\mathcal{F}}_{mn})^2 \right] + \dots , \tag{2.16}$$

where $\tilde{V}_{2l}$ is the volume of the dual torus,

$$V_{2l}\tilde{V}_{2l} = (\frac{2\pi}{T})^{2l} = (2\pi)^{2l} . \tag{2.17}$$

The $\tilde{\mathcal{F}}^4$ coefficient in this expression is exactly the same as the quartic term in the expansion of the Born-Infeld action $\sqrt{\det(\delta_{mn} + \tilde{\mathcal{F}}_{mn})}$ or in the open string effective

[3] Let us note that the subleading $\frac{1}{r^{2(7-p)}}$ term in $\mathcal{V}$ (2.9) is proportional to (cf. (2.10))

$$\prod_{m=1}^{l} \sqrt{1+f_m^2} \left[\sum_{k=1}^{l} \frac{1}{2(1+f_k^2)} + \prod_{k=1}^{l} \frac{f_k}{\sqrt{1+f_k^2}} - \frac{1}{8}(\sum_{k=1}^{l} \frac{1}{1+f_k^2})^2 + \frac{1}{4}\sum_{k=1}^{l} \frac{1}{(1+f_k^2)^2} \right].$$

The leading term in the large field ($f_k \to \infty$) expansion of this expression vanishes.

[4] D-instanton does not couple to D-string charge; the contribution of the latter is in any case suppressed for large f_k.

action. This can be seen directly from (2.8) by noting that the expression there is $H_{-1}^{-1}[\sqrt{\det(G_{mn}+\mathcal{F}_{mn})}-\sqrt{\det\mathcal{F}_{mn}}]=\sqrt{\det\mathcal{F}_{mn}}\Big(H_{-1}^{-1}[\sqrt{\det(\delta_{mn}+H_{-1}^{1/2}\mathcal{F}_{mn}^{-1})}-1]\Big)$. The reason for this non-trivial coincidence (note that $\tilde{\mathcal{F}}_{mn}$ is the *inverse* of the background field $\mathcal{F}_{nm}$ in the probe action) will become clear below when we reproduce (2.16) from the matrix model.

2.2 Interaction of 'D-string' with 3-brane–instanton bound state

To determine the interaction potential between the non-marginal bound state of D-string and D-instanton and the marginal bound state of D3-brane and D-instanton we shall consider $1+i$ as a probe moving in the $3\|i$ background. As above, the action for the $1+i$ probe will be the D-string action (2.1) with a constant flux (2.3) on 2-torus representing the D-instanton charge.

The $3\|i$ type IIB supergravity background [16] is T-dual to $4\|0$ or $5\|1$ solutions [27]. We shall assume that the 3-brane world volume is wrapped around 4-torus (in directions $1,2,3,4$) and that the world volume of $(1+i)$-brane probe is parallel to $(5,6)$ directions, i.e. that the world volumes do not share common dimensions.[5] The corresponding metric, dilaton and RR scalar fields smeared in the $(5,6)$ directions are [27]

$$ds^2_{10}=(H_{-1}H_3)^{1/2}[H_3^{-1}(dy_1^2+...+dy_4^2)+dy_5^2+dy_6^2+dx_idx_i]\ , \tag{2.18}$$

$$e^\phi=H_{-1}\ ,\qquad C_0=H_{-1}^{-1}-1\ ,\qquad H_{-1}=1+\frac{Q^{(6)}_{-1}}{r^2}\ ,\qquad H_3=1+\frac{Q^{(2)}_3}{r^2}\ ,$$

where $Q^{(n)}_p$ are given by (2.7) ($C_2=0$; the value of C_4 background will not be important below). Ignoring the dependence on derivatives of X_i we find for the 'D-string' probe action I_1 ($f\equiv f_1$)

$$I_1=-T_1\int d^2x\Big[H_{-1}^{-1}\sqrt{H_{-1}H_3+f^2}-(H_{-1}^{-1}-1)f\Big]$$
$$=-T_1V_2f\Big[1+H_{-1}^{-1}\Big(\sqrt{1+H_{-1}H_3f^{-2}}-1\Big)\Big]\equiv-T_1V_2\sqrt{1+f^2}-\mathcal{V}\ . \tag{2.19}$$

The leading long-distance interaction term in $\mathcal{V}$ is

$$\mathcal{V}=\frac{1}{2r^2}T_1V_2\sqrt{1+f^2}\Big[Q_3^{(2)}\frac{1}{1+f^2}-Q_{-1}^{(6)}\Big(1-\frac{f}{\sqrt{1+f^2}}\Big)^2\Big]+O(\frac{1}{r^4})\ . \tag{2.20}$$

This expression is in direct T-duality correspondence with the static potential between the $2+0$ and $4\|0$ bound states in [18].

The potential (2.20) has the following large f (large instanton charge n_{-1} of $1+i$) expansion, cf.(2.13)

$$\mathcal{V}=\frac{1}{2r^2}n_1\Big(N_3\tilde f-N_{-1}\pi^2V_4^{-1}\tilde f^3-\tfrac12N_3\tilde f^3\Big)+...\ ,\qquad \tilde f\equiv f^{-1}\ , \tag{2.21}$$

[5] Here the adequate interpretation is that the time direction is orthogonal to both of the $(1+i)$ and $(3\|i)$ world-volumes [15].

where V_4 is the volume of the 4-torus.[6] $\mathcal{V}$ can be expressed in terms of

$$n_{-1} = n_1 (2\pi)^{-1} V_2 f = n_1 2\pi \tilde{V}_2^{-1} \tilde{f}^{-1} \tag{2.22}$$

as follows

$$\mathcal{V} = \frac{\tilde{V}_2}{4\pi r^2}\, n_{-1} \left(N_3 \tilde{f}^2 - \tfrac{1}{(4\pi)^2} N_{-1} \tilde{V}_4 \tilde{f}^4 - \tfrac{1}{2} N_3 \tilde{f}^4 \right) + \ldots\ . \tag{2.23}$$

Since in the matrix model representation N_3 will be the instanton number of a gauge field on the dual 4-torus, (2.23) will be, like (2.16), proportional to the integral of F^4 terms over the dual 6-torus ($N_3 \tilde{f}^4$ will be a subleading correction).

3 One-loop effective action in $D \leq 10$ super Yang-Mills theory

To put matrix model computations in a proper perspective, it is useful to give a summary of some general results about the one-loop effective action $\Gamma(A)$ of maximally supersymmetric YM theory in $D \leq 10$ dimensions.

3.1 UV divergences and 'large mass' expansion

In general,

$$\Gamma = \tfrac{1}{2} \sum_a c_a \ln \det \Delta_a = -\tfrac{1}{2} \int_{\Lambda^{-2}}^{\infty} \frac{ds}{s}\ \mathrm{tr} \sum_a c_a e^{-s\Delta_a}\ , \tag{3.1}$$

where the sum over a runs over bosonic, background gauge ghost and fermionic contributions taken with appropriate relative coefficients ($c_a = 1, -2, -\frac{1}{4}$). Δ_a are second order differential operators $(-D^2 + \mathcal{X})$ depending on background value of the gauge field and $\Lambda \to \infty$ is UV cutoff. The divergent part of Γ can be expressed in terms of the DeWitt-Seeley coefficients $\mathbf{b}_n$

$$(\mathrm{tr}\ e^{-s\Delta})_{s\to 0} \simeq \frac{1}{(4\pi)^{D/2}} \sum_{n=0}^{\infty} s^{\frac{n-D}{2}} \int d^D x\ \mathbf{b}_n(\Delta)\ , \tag{3.2}$$

i.e.

$$\Gamma^{(\infty)} = -\frac{1}{(4\pi)^{D/2}} \int d^D x \Big(\frac{\Lambda^D}{D}\ \mathbf{b}_0 + \frac{\Lambda^{D-2}}{D-2}\ \mathbf{b}_2 + \frac{\Lambda^{D-4}}{D-4}\ \mathbf{b}_4 + \ldots + \frac{1}{2} \ln \Lambda^2\ \mathbf{b}_D \Big)\ , \tag{3.3}$$

where $\mathbf{b}_n \equiv \sum_a c_a \mathbf{b}_n(\Delta_a)$. For pure YM theory [28] $\mathbf{b}_4 = \frac{1}{12}(D-26)\mathrm{Tr} F_{mn}^2$ (the appearance of the coefficient $D-26$ can be understood from string theory [29]), while for $D = 10$ SYM theory and its reductions to lower dimensions [28]

$$\mathbf{b}_0 = \mathbf{b}_2 = \mathbf{b}_4 = \mathbf{b}_6 = 0\ , \tag{3.4}$$

[6]The large f limit of the $\frac{1}{r^4}$ subleading term in $\mathcal{V}$ is $-\frac{1}{8r^4} T_1 V_2 Q_3^{(2)} (2Q_{-1}^{(6)} + Q_3^{(2)}) f^{-3}$.

so that SYM theories in $D \leq 7$ are one-loop UV finite. At the same time, $\mathbf{b}_8$ and $\mathbf{b}_{10}$ are, in general, non-vanishing. In particular, in a constant abelian background $\mathbf{b}_{10} = 0$ but $\mathbf{b}_8 \sim F^4 \neq 0$, implying the presence of logarithmic divergence in $D = 8$ SYM and quadratic divergence in $D = 10$ theory [28]. The general non-abelian expressions for $\mathbf{b}_8$ and $\mathbf{b}_{10}$ (up to F^5 terms) in SYM theory were found in [29] (basing on the results of [30])

$$\mathbf{b}_8 = \tfrac{2}{3}\mathrm{Tr}\Big(F_{mk}F_{nk}F_{mr}F_{nr} + \tfrac{1}{2}F_{mk}F_{nk}F_{nr}F_{mr} - \tfrac{1}{4}F_{mk}F_{mk}F_{nr}F_{nr} - \tfrac{1}{8}F_{mk}F_{nr}F_{mk}F_{nr}\Big) , \tag{3.5}$$

$$\mathbf{b}_{10} = -\tfrac{1}{15}\mathrm{Tr}\Big(D_qF_{mk}F_{nk}D_qF_{mr}F_{nr} + \tfrac{1}{2}D_qF_{mk}F_{nk}D_qF_{nr}F_{mr}$$
$$- \tfrac{1}{4}D_qF_{mk}F_{mk}D_qF_{nr}F_{nr} - \tfrac{1}{8}D_qF_{mk}F_{nr}D_qF_{mk}F_{nr}\Big) + O(F^5) . \tag{3.6}$$

The trace Tr is in the adjoint representation[7] and we dropped gauge-dependent $O(D_mF_{mk})$ terms which vanish on the equations of motion.

The reason why the structure of $\mathbf{b}_8$ (i.e. of the coefficient of quadratic divergence in $D = 10$ SYM) is the same as of the F^4 term in the open superstring effective action was explained in [29].[8]

Let us now formally shift Δ_a by the same constant term M^2 and define 'IR regularised' effective action Γ_M

$$\Gamma_M \equiv \tfrac{1}{2}\sum_a c_a \ln\det(\Delta_a + M^2) = -\tfrac{1}{2}\int_{\Lambda^{-2}}^{\infty} \frac{ds}{s} e^{-sM^2}\,\mathrm{tr}\sum_a c_a e^{-s\Delta_a} . \tag{3.7}$$

This modified 1-loop effective action is finite in $D \leq 7$ and has the following *large M* expansion (we use (3.2))

$$\Gamma_M \simeq -\tfrac{1}{2}\int d^Dx \sum_{n=0}^{\infty} \frac{\Gamma(\frac{n-D}{2})}{(4\pi)^{D/2}M^{n-D}}\ \mathbf{b}_n = -\frac{1}{2(4\pi)^{D/2}}\int d^Dx\Big[\frac{\Gamma(\frac{8-D}{2})}{M^{8-D}}\ \mathbf{b}_8 + \frac{\Gamma(\frac{10-D}{2})}{M^{10-D}}\ \mathbf{b}_{10} + ...\Big]. \tag{3.8}$$

The explicit form of the leading term is

$$\Gamma_M = -\frac{(3-\frac{1}{2}D)!}{3(4\pi)^{D/2}M^{8-D}}\int d^Dx\ \mathrm{Tr}\Big(F_{mk}F_{nk}F_{mr}F_{nr} + \tfrac{1}{2}F_{mk}F_{nk}F_{nr}F_{mr}$$
$$- \tfrac{1}{4}F_{mk}F_{mk}F_{nr}F_{nr} - \tfrac{1}{8}F_{mk}F_{nr}F_{mk}F_{nr}\Big) + O(\frac{1}{M^{10-D}}) , \tag{3.9}$$

[7]For generators of $SU(N)$ $\mathrm{Tr}(T_aT_b) = N\delta_{ab}$, $\mathrm{tr}(T_aT_b) = \frac{1}{2}\delta_{ab}$ and $\mathrm{Tr}X^2 = 2N\mathrm{tr}X^2$, $\mathrm{Tr}X^4 = 2N\mathrm{tr}X^4 + 6(\mathrm{tr}X^2)^2$, $X = X^aT_a$ (see [31]; similar expressions in Appendix B of [29] should be multiplied by factor of 2). The same relations are true for a matrix X belonging to $U(N)$ algebra provided X in the r.h.s. is replaced by its traceless part $X \to \bar{X} = X - \frac{1}{N}\mathrm{tr}X\ \mathrm{I}$.

[8] This term can be extracted from the $\alpha' \to 0$ limit of the string one-loop effective action ($\frac{1}{\alpha'}F^4 \to \Lambda^2F^4$) if one includes planar as well as non-planar ($(\mathrm{tr}F^2)^2$) contributions. The tree-level open string effective action contains similar F^4 term (the kinematic factor in the tree-level and 1-loop 4-vector amplitude is the same [33]) but with tr instead of Tr [32].

or, equivalently, in terms of the trace in the fundamental representation of $U(N)$

$$\Gamma_M = -\frac{2(3-\frac{1}{2}D)!}{3(4\pi)^{D/2}M^{8-D}}\int d^Dx\Big(\ N\ \mathrm{tr}\Big[\bar F_{mk}\bar F_{nk}\bar F_{mr}\bar F_{nr}+\tfrac{1}{2}\bar F_{mk}\bar F_{nk}\bar F_{nr}\bar F_{mr}$$
$$-\tfrac{1}{4}\bar F_{mk}\bar F_{mk}\bar F_{nr}\bar F_{nr}-\tfrac{1}{8}\bar F_{mk}\bar F_{nr}\bar F_{mk}\bar F_{nr}\Big]$$
$$+\ 3\Big[\mathrm{tr}(\bar F_{mk}\bar F_{nk})\ \mathrm{tr}(\bar F_{mr}\bar F_{nr})+\tfrac{1}{2}\mathrm{tr}(\bar F_{mk}\bar F_{nr})\ \mathrm{tr}(\bar F_{nk}\bar F_{mr})$$
$$-\ \tfrac{1}{4}\mathrm{tr}(\bar F_{mk}\bar F_{nr})\ \mathrm{tr}(\bar F_{mk}\bar F_{nr})-\tfrac{1}{8}\mathrm{tr}(\bar F_{mk}\bar F_{mk})\ \mathrm{tr}(\bar F_{nr}\bar F_{nr})\Big]\Big)+O(\frac{1}{M^{10-D}})\ , \qquad (3.10)$$

where $\bar F_{mn}\equiv F_{mn}-\frac{1}{N}\mathrm{tr}F_{mn}\mathrm{I}$. This expansion is useful in discussions of long-distance interactions between Dp-branes where $D=p+1$ and M is proportional to separation b between branes, i.e. $M^2=Tb^2$ (expressions related to special cases of (3.9),(3.10) appeared in [1, 6, 34] and, in particular, in [20]; see also below).

Note that the subleading $O(\frac{1}{M^{10-D}})$ correction determined by $\mathbf{b}_{10}$ vanishes in the case of constant abelian backgrounds which describe, e.g., interactions between 1/2 supersymmetric non-marginal bound states of D-branes. The coefficient $\mathbf{b}_{10}$ is, in general, non-vanishing for non-abelian background fields.

3.2 Constant abelian gauge field background

The one-loop effective action of SYM theory in D dimensions can be computed explicitly for a constant abelian gauge field background (i.e. for $F_{mn}=\mathrm{F}^I_{mn}T_I$ belonging to the Cartan subalgebra of a compact semisimple Lie algebra) following [28, 35]. The basis T_I $(I=1,...,r)$ in the Cartan subalgebra in the adjoint representation can be chosen as a set of $d\times d$ diagonal matrices $T_I=\mathrm{diag}(0,...,0,\alpha_I^{(1)},-\alpha_I^{(1)},...,\alpha_I^{(q)},-\alpha_I^{(q)})$, where $\{\alpha_I^{(i)}\}$ are positive roots $(i=1,...,q,\ q=\frac{1}{2}(d-r),\quad \sum_{i=1}^q\alpha_I^{(i)}\alpha_{I'}^{(i)}=\delta_{II'})$. Let us define $\mathrm{F}^{(i)}_{mn}=\mathrm{F}^I_{mn}\alpha_I^{(i)}$ and assume that all $\mathrm{F}^{(i)}_{mn}$ have 'block-diagonal' form (we choose space-time dimension to be even $D=2l$)[9]

$$\mathrm{F}^{(i)}_{mn}=\begin{pmatrix} 0 & \mathrm{f}_1^{(i)} & & & \\ -\mathrm{f}_1^{(i)} & 0 & & & \\ & & \ddots & & \\ & & & 0 & \mathrm{f}^{(i)}_{D/2} \\ & & & -\mathrm{f}^{(i)}_{D/2} & 0\end{pmatrix}, \qquad (3.11)$$

Then one finds the following general expression for Γ_M in (3.7) $(\mathrm{V}_D\equiv\int d^Dx)$

$$\Gamma_M=-\frac{2\mathrm{V}_D}{(4\pi)^{D/2}}\int_0^\infty\frac{ds}{s^{1+D/2}}\,e^{-M^2s}\ \sum_{i=1}^q\Big(\prod_{k=1}^{D/2}\frac{\mathrm{f}_k^{(i)}s}{\sinh\mathrm{f}_k^{(i)}s}$$

[9] The expressions that follow are true also in more general case if the parameters $\mathrm{f}_k^{(i)}$ are simply replaced by Lorentz invariants constructed out of $\mathrm{F}^{(i)}_{mn}$ (separately for each i) according to the rules [35]: $\sum_{k=1}^l(\mathrm{f}_k^{(i)})^{2h}=\frac{1}{2}(-1)^h\mathrm{F}^{(i)}_{m_1n_1}\mathrm{F}^{(i)}_{n_1m_2}...\mathrm{F}^{(i)}_{n_{h-1}m_1},\quad h=1,...,l.$

$$\times \Big[\sum_{k=1}^{D/2} \Big(\cosh 2\mathrm{f}_k^{(i)} s - 1 \Big) - 4 (\prod_{k=1}^{D/2} \cosh \mathrm{f}_k^{(i)} s - 1) \Big] \Big) \ . \tag{3.12}$$

In what follows we shall consider the special case when the background is such that $\mathbf{N}$ of $\mathrm{F}^{(i)}_{mn}$ are equal to the same F_{mn} while the rest vanish, i.e. when $\mathrm{f}_k^{(i)} = \mathrm{f}_k, \ \ i = 1, ..., \mathbf{N}$. The corresponding background field strength is given by diagonal matrices in the adjoint or fundamental representaions:

$$F^{(adj)}_{mn} = \mathrm{diag}(0, ..., 0, \mathrm{F}_{mn}, -\mathrm{F}_{mn}, ..., \mathrm{F}_{mn}, -\mathrm{F}_{mn}) \ , \qquad F^{(fund)}_{mn} = \begin{pmatrix} \mathrm{F}_{mn} \ \mathrm{I} & 0 \\ 0 & 0 \end{pmatrix} \ ,$$

where I a unit $n \times n$ matrix and $\mathbf{N} = n(N - n)$. Then

$$\Gamma_M = - \frac{2\mathbf{N} \ \mathrm{V}_D}{(4\pi)^{D/2}} \int_0^\infty \frac{ds}{s^{1+D/2}} \, e^{-M^2 s} \ \prod_{k=1}^{D/2} \frac{\mathrm{f}_k s}{\sinh \mathrm{f}_k s} \ \Big[\sum_{k=1}^{D/2} (\cosh 2\mathrm{f}_k s - 1) - 4 (\prod_{k=1}^{D/2} \cosh \mathrm{f}_k s - 1) \Big] . \tag{3.13}$$

This integral is UV convergent for $D \leq 7$ and logarithmically divergent for $D = 8$ implying also the presence of $O(F^4)$ quadratic UV divergence in $D = 10$ SYM theory. It is also IR divergent for certain $\mathrm{f}_k^{(i)}$ and small enough M (which is a manifestation of the well-known tachyonic instability of the YM theory in a constant abelian background which is not cured by supersymmetry).

For example, the standard ($M = 0$) one-loop effective action for maximally supersymmetric $SU(2)$ YM theory in $D = 4$ in background $F_{mn} = \mathrm{F}_{mn} \frac{\sigma_3}{2}$ (i.e. $\mathbf{N} = 1$) is [28]

$$\Gamma = - \frac{4\mathrm{V}_4}{(4\pi)^2} \int_0^\infty \frac{ds}{s^3} \ \frac{\mathrm{f}_1 s}{\sinh \mathrm{f}_1 s} \ \frac{\mathrm{f}_2 s}{\sinh \mathrm{f}_2 s} \ (\cosh \mathrm{f}_1 s - \cosh \mathrm{f}_2 s)^2 \ . \tag{3.14}$$

For comparison with the matrix model expressions, it is useful to separate a factor $\mathcal{N} \sim \sqrt{\det \, \mathrm{F}_{mn}}$ in Γ_M representing it as

$$\Gamma_M = \mathbf{N} \, \mathcal{N} \, \mathcal{W} \ , \tag{3.15}$$

$$\mathcal{N} \equiv (2\pi)^{-D/2} \mathrm{V}_D \prod_{k=1}^{D/2} \mathrm{f}_k = (2\pi)^{-D/2} \mathrm{V}_D \sqrt{\det \, \mathrm{F}_{mn}} \ , \tag{3.16}$$

$$\mathcal{W} = -2 \int_0^\infty \frac{ds}{s} \, e^{-M^2 s} \ \prod_{k=1}^{D/2} \frac{1}{2 \sinh \mathrm{f}_k s} \ \Big[\sum_{k=1}^{D/2} (\cosh 2\mathrm{f}_k s - 1) - 4 (\prod_{k=1}^{D/2} \cosh \mathrm{f}_k s - 1) \Big] \ . \tag{3.17}$$

In the matrix model context $\mathcal{N}^{-1}$ will be an integer (or a rational number, cf. (2.15),(2.22)) and will be cancelled against a factor contained in $\mathbf{N}$ (see section 4).

Special cases of Γ_M (3.15) or (up to an overall coefficient) $\mathcal{W}$ appeared in the discussions of interaction potentials between D-branes (see, e.g., [36, 1, 17, 11, 12]).[10] The

[10] For example, for $\mathrm{f}_1 = iv, \ \mathrm{f}_2, ..., \mathrm{f}_l = 0, \ \ M = b$ we get from (3.17) the (light open string mode part of) phase shift for the scattering of two 0-branes [36], $\delta = -i\mathcal{W} = \int_0^\infty \frac{ds}{s} \, e^{-b^2 s} (\sin vs)^{-1} (\cos 2vs - 4 \cos vs + 3)$.

general expression (3.17) was given in [6], where it was describing the potential between parallel 'Dp-brane' and anti-'Dp-brane'.

The leading terms in the large M expansions of Γ_M and $\mathcal{W}$ can be found directly from (3.13),(3.17)

$$\Gamma_M = -\frac{(3-\frac{1}{2}D)!\ \mathbf{N}\ V_D}{(4\pi)^{D/2}M^{8-D}}\ \Big[2\sum_{k=1}^{D/2}\mathrm{f}_k^4 - (\sum_{k=1}^{D/2}\mathrm{f}_k^2)^2\Big] + O(\frac{1}{M^{10-D}})$$

$$= -\frac{(3-\frac{1}{2}D)!\ \mathbf{N}\ V_D}{(4\pi)^{D/2}M^{8-D}}\ \Big[\mathrm{F}_{mk}\mathrm{F}_{nk}\mathrm{F}_{mr}\mathrm{F}_{nr} - \frac{1}{4}(\mathrm{F}_{mk}\mathrm{F}_{mk})^2\Big] + O(\frac{1}{M^{10-D}})\ , \qquad (3.18)$$

$$\mathcal{W} = -\frac{(3-\frac{1}{2}D)!}{2^{D/2}M^{8-D}}\ \frac{1}{\sqrt{\det\ \mathrm{F}_{mn}}}\ \Big[\mathrm{F}_{mk}\mathrm{F}_{nk}\mathrm{F}_{mr}\mathrm{F}_{nr} - \frac{1}{4}(\mathrm{F}_{mk}\mathrm{F}_{mk})^2\Big] + O(\frac{1}{M^{10-D}})\ . \qquad (3.19)$$

They have the expected F^4 structure (3.9). In fact, for the abelian background considered above one has from (3.5):

$$\mathbf{b}_8 = \mathrm{Tr}[F^4 - \frac{1}{4}(F^2)^2] = 2\mathbf{N}\ [\mathrm{F}^4 - \frac{1}{4}(\mathrm{F}^2)^2] = 2\mathbf{N}\ \Big[2\sum_{k=1}^{D/2}\mathrm{f}_k^4 - (\sum_{k=1}^{D/2}\mathrm{f}_k^2)^2\Big]\ . \qquad (3.20)$$

4 Matrix model (super Yang-Mills) description

In this section we shall demonstrate that the leading-order terms in the long-distance potentials between BPS bound states with 1/2 and 1/4 of supersymmetry (2.12) and (2.21) computed in section 2 using classical closed string effective field theory methods are indeed reproduced by the instanton matrix model, i.e. by the corresponding 1-loop SYM computations.

The instanton matrix model is defined by the $D = 10$ $U(N)$ SYM Lagrangian reduced to 0+0 dimensions (in this section we shall assume that $T^{-1} = 2\pi\alpha' = 1$)

$$L = -\frac{1}{2g_s}\mathrm{tr}\Big(\frac{1}{2}\,[X_\mu, X_\nu]^2 + 2\theta^T\gamma_\mu\,[\theta, X_\mu]\Big)\,, \qquad (4.1)$$

where the elements of $N \times N$ matrix θ are 16-component real spinors and $\gamma_{10} \equiv \mathrm{I}_{16\times 16}$.

We shall consider the background gauge field $\bar{A}_\mu = T\left(\bar{X}_1, \ldots, \bar{X}_{10}\right)$ where the components

$$\bar{X}_i = \begin{pmatrix} \bar{X}_i^{(1)} & 0 \\ 0 & \bar{X}_i^{(2)} \end{pmatrix}, \qquad i = 1, \ldots, 8, 10\ , \qquad (4.2)$$

correspond to the coordinates of the two BPS objects and

$$\bar{X}_9 = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \qquad (4.3)$$

represents their separation b. The calculation of the SYM one-loop effective action in this background is similar to the one described in [18]. Let us define the operators

$$H = \left(\bar{X}_i^{(1)} \otimes \mathrm{I} - \mathrm{I} \otimes \bar{X}_i^{(2)*}\right)^2 , \qquad H_{ij} = \bar{X}_{ij}^{(1)} \otimes \mathrm{I} + \mathrm{I} \otimes \bar{X}_{ij}^{(2)*} , \tag{4.4}$$

where $\bar{X}_{ij}^{(1)} = [\bar{X}_i^{(1)}, \bar{X}_j^{(1)}]$, $\bar{X}_{ij}^{(2)} = [\bar{X}_i^{(2)}, \bar{X}_j^{(2)}]$ and $*$ is the complex conjugation. The 1-loop effective action is the sum of the bosonic, ghost and fermionic contributions,

$$W = W_B + W_G + W_F , \tag{4.5}$$

$$W_B = \ln\det(H\delta_{\mu\nu} + 2H_{\mu\nu}) , \qquad W_G = -2\ln\det\, H , \qquad W_F = -\tfrac{1}{2}\ln\det(H + \sum_{i<j}\gamma_i\gamma_j H_{ij}) ,$$

where the operators act in the $U(N)$ matrix index space, Lorentz vector space and Lorentz spinor space. In the case of the background

$$\bar{X}_{10} = \begin{pmatrix} i\partial_\tau & 0 \\ 0 & 0 \end{pmatrix} , \qquad \bar{X}_{i_*} = \begin{pmatrix} v\tau & 0 \\ 0 & 0 \end{pmatrix} ,$$

the resulting expression for W in (4.5) becomes the same as found in the 0-brane matrix model [18] for the relative motion of two BPS objects along the direction i_*. This may be viewed as a manifestation of T-duality in string theory or Eguchi-Kawai reduction in large N SYM theory [37, 9, 1].

4.1 D-instanton – ‘Dp-brane’ interaction

A ‘Dp-brane’ wrapped over a torus $\tilde{T}^{p+1}$ is represented by the following classical solution of the instanton matrix model ($m, n = 1, ..., p+1 = 2l$)

$$\bar{X}_m = T^{-1}(i\partial_m + \tilde{A}_m)\mathrm{I}_{n_{-1}\times n_{-1}} , \qquad [\bar{X}_m, \bar{X}_n] = iT^{-2}\tilde{F}_{mn}\mathrm{I}_{n_{-1}\times n_{-1}} , \tag{4.6}$$

where ∂_m act on functions on the torus and $\tilde{F}_{mn}$ is a constant abelian field strength. This configuration corresponds [3] to the $(p+(p-2)+\cdots+1+i)$ type IIB bound state wrapped over the torus T^{p+1} dual to $\tilde{T}^{p+1}$. We shall choose $\tilde{F}_{mn}$ in the form

$$\tilde{\mathcal{F}}_{mn} \equiv T^{-1}\tilde{F}_{mn} = \begin{pmatrix} 0 & \tilde{f}_1 & & & \\ -\tilde{f}_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \tilde{f}_l \\ & & & -\tilde{f}_l & 0 \end{pmatrix} . \tag{4.7}$$

This background field is (minus) the inverse of the one which appears in the T-dual string theory picture (2.3), i.e. $\tilde{\mathcal{F}}_{mn}\mathcal{F}_{m'n} = \delta_{mm'}$, or $\tilde{f}_k = f_k^{-1}$.

Let us explain the reason for this inverse identification between the fluxes in the matrix model and string theory descriptions (see [9, 11] for discussions of T-dual type IIA cases).

The $U(N)$ SYM theory on T^{p+1}represents $n_p = N$ Dp-branes with euclidean world-volumes wrapped over the torus [13]. By T-duality [37], it is also describing $\tilde{n}_{-1} = N$ D-instantons on the dual torus $\tilde{T}^{p+1}$. Turning on the background field (2.3) on T^{p+1} we get a non-marginal bound state $p + (p-2) + ... + i$ with the 'induced' instanton number (2.4),(2.15) equal to $n_{-1} = n_p V_{p+1} (\frac{Tf}{2\pi})^{\frac{p+1}{2}}$ (for simplicity here we set all f_k to be equal to f). Since T-duality along all of the directions of the torus T^{p+1} interchanges instantons with Dp-branes, the corresponding bound state wrapped over $\tilde{T}^{p+1}$ contains $\tilde{n}_p = n_{-1}$ Dp-branes and $\tilde{n}_{-1} = n_p$ instantons. If the background field $\tilde{\mathcal{F}}_{mn}$ that produces this charge distribution is (4.7), then $\tilde{n}_{-1} = \tilde{n}_p \tilde{V}_{p+1} (\frac{T\tilde{f}}{2\pi})^{\frac{p+1}{2}}$. As a result,

$$\tilde{n}_p = n_{-1} \ , \qquad \tilde{n}_{-1} = n_p \tag{4.8}$$

implies

$$V_{p+1} (\frac{Tf}{2\pi})^{\frac{p+1}{2}} \ \tilde{V}_{p+1} (\frac{T\tilde{f}}{2\pi})^{\frac{p+1}{2}} = 1 \quad , \qquad \text{i.e.} \qquad f \ \tilde{f} = 1 \ , \tag{4.9}$$

where we used (2.17).

The matrix model background describing the configuration of 'Dp-brane' ($p = 2l - 1$) with the world-volume directions $X_1, ..., X_{p+1}$ and N_{-1} D-instantons located at the origin, which are separated from each other by a distance b in the 9-th direction is thus represented by

$$\bar{X}^{(1)}_1 = q_1 \ , \quad \bar{X}^{(1)}_2 = p_1 \ , \ ... \ , \qquad \bar{X}^{(1)}_{2l-1} = q_l \ , \quad \bar{X}^{(1)}_{2l} = p_l \ , \qquad \bar{X}^{(1)}_9 = b \ , \tag{4.10}$$

$$[q_k, p_n] = i\tilde{f}_k \delta_{kn} \mathrm{I} \ , \qquad \tilde{f}_k = f_k^{-1} \ , \tag{4.11}$$

with all other $\bar{X}^{(1)}_\mu$ components being equal to zero and the $N_{-1} \times N_{-1}$ matrix $\bar{X}^{(2)}_\mu$ ($\mu = 1, ..., 10$) having zero entries.

The bosonic, ghost and fermionic contributions to the 1-loop effective action W (4.5) in this background are

$$W_B = N_{-1} \sum_{\{n\}=0}^{\infty} \mathrm{Tr} \ \ln \left(b^2_{\{n\}} \delta_{\mu\nu} - 2i\tilde{\mathcal{F}}_{\mu\nu} \right) \ , \qquad W_G = -2N_{-1} \sum_{\{n\}=0}^{\infty} \mathrm{Tr} \ \ln \ b^2_{\{n\}} \ , \tag{4.12}$$

$$W_F = -\tfrac{1}{2} N_{-1} \sum_{\{n\}=0}^{\infty} \mathrm{Tr} \ \ln \left(b^2_{\{n\}} + \frac{i}{2} \gamma_{ij} \tilde{\mathcal{F}}_{ij} \right) , \qquad b^2_{\{n\}} \equiv b^2 + \sum_{k=1}^{l} \tilde{f}_k (2n_k + 1) \ , \tag{4.13}$$

where $\{n\} \equiv \{n_1, ..., n_l\}$. The constant background field matrix $\tilde{\mathcal{F}}_{\mu\nu}$ has (4.7) as non-zero entries. The resulting effective action W (4.5) is given by

$$W = -2N_{-1} \int_0^{\infty} \frac{ds}{s} \, e^{-b^2 s} \prod_{k=1}^{l} \frac{1}{2 \sinh \tilde{f}_k s} \Big[\sum_{k=1}^{l} \left(\cosh 2\tilde{f}_k s - 1 \right) - 4 (\prod_{k=1}^{l} \cosh \tilde{f}_k s - 1) \Big] \ . \tag{4.14}$$

This is equal to the 1-loop effective action Γ_M of the $U(N)$ SYM theory on the dual torus $\tilde{T}^{p+1}$ in a constant abelian background proportional to $\tilde{\mathcal{F}}_{mn}$ and with an IR cutoff $M = b$ (see (3.7),(3.13),(3.15)). The dimension of the fundamental representation of the

YM gauge group is the total number of instantons $N = N_{-1} + n_{-1}$ (with both N_{-1} and n_{-1} assumed to be large).

Indeed, let us set $D = p+1 = 2l$, $\mathrm{f}_k = \tilde{f}_k$, $\mathrm{V}_D = \tilde{V}_{2l}$ and

$$N = N_{-1} + n_{-1} \ , \qquad \mathbf{N} = n_{-1}N_{-1} \tag{4.15}$$

in the SYM expression (3.13) or (3.15),(3.17). The corresponding abelian $U(N)$ SYM background in the fundamental representation is given by a diagonal $N \times N$ matrix $F_{mn} = \begin{pmatrix} \tilde{\mathcal{F}}_{mn}\,\mathrm{I} & 0 \\ 0 & 0 \end{pmatrix}$ where I is a unit $n_{-1} \times n_{-1}$ matrix. In the adjoint representantion it has $\mathbf{N} = n_{-1}N_{-1}$ non-zero entries $(\tilde{\mathcal{F}}_{mn},\ -\tilde{\mathcal{F}}_{mn})$ (differences of diagonal values of the Cartan subalgebra element in the fundamental representation). Equivalently, $\mathbf{N} = q(N_{-1}+n_{-1}) - q(N_{-1}) - q(n_{-1}) = n_{-1}N_{-1}$, where $q(N) = \frac{1}{2}N(N-1)$ is the number of positive roots of $U(N)$. The resulting SYM effective action is thus given by (3.13).

Since the factor $\mathcal{N}$ in (3.16) on the dual torus is $\mathcal{N} = \frac{\tilde{n}_{-1}}{\tilde{n}_p} = \frac{n_p}{n_{-1}}$, we conclude that for $n_p = 1$ (assumed in the derivation of (4.14)) the factor of $\mathcal{N} = \frac{1}{n_{-1}}$ cancels out, i.e.

$$\Gamma_M = \mathcal{N}\ \mathbf{N}\ \mathcal{W} = N_{-1}\ \mathcal{W} = W \ . \tag{4.16}$$

Retaining only the leading term in the large distance ($b \to \infty$) expansion of W, we find the same expression as in (3.18),(3.19)

$$W = -\frac{1}{b^{8-2l}} 2^{-l}(3-l)!\, N_{-1} \prod_{k=1}^{l} \tilde{f}_k^{-1} \left[2\sum_{k=1}^{l} \tilde{f}_k^4 - (\sum_{k=1}^{l} \tilde{f}_k^2)^2 \right] + O(\frac{1}{b^{10-2l}}) \, . \tag{4.17}$$

Remarkably, with $b = r$ and $\tilde{f}_k = f_k^{-1}$ this coincides with the long-distance interaction potential (2.12),(2.14),(2.16) found from supergravity in the limit of *large* instanton number n_{-1} (large f_k or small $\tilde{f}_k$).

The coefficient of the subleading $\frac{1}{b^{10-2l}}$ term in (4.17) turns out to be zero. This is a consequence of the vanishing of the coefficient $\mathbf{b}_{10}$ (3.6) in (3.8) in a constant abelian background. Note, however, that the powers of $r = b$ in the subleading terms in (4.14),(4.17) and in the supergravity expression (2.10) do not match in general.

The same universal expressions (4.17) or (3.20) describe also interactions of T-dual configurations of branes in the 0-brane matrix model. For example, the scattering of the two 0-branes is represented by the (electric) background $F_{01} = iv$, $N = n_0 + N_0$, $\mathbf{N} = n_0 N_0$, i.e. $l = 1$, $\tilde{f}_1 = iv$, $\delta = -iW \sim \frac{1}{r^6}v^3$. The case of a 0-brane scattering on a $2+0$ brane is represented by $N \times N$ matrix $F_{mn} = \begin{pmatrix} \tilde{\mathcal{F}}^{(1)}_{mn}\,\mathrm{I}_{n_0\times n_0} & 0 \\ 0 & \tilde{\mathcal{F}}^{(2)}_{mn}\,\mathrm{I}_{N_0\times N_0} \end{pmatrix}$, where $\tilde{\mathcal{F}}^{(1)}_{mn} = \tilde{f}\epsilon_{mn}$ for $m,n = 2,3$, $\tilde{\mathcal{F}}^{(2)}_{mn} = iv\epsilon_{mn}$ for $m,n = 0,1$ (1 is dual to the direction of the 0-brane motion, $2,3$ are dual to the directions of 2-brane) and we assume that the volume $\tilde{V}_2$ of the two-torus $(\tilde{x}_0, \tilde{x}_1)$ is chosen so that $iv\tilde{V}_2 = 2\pi$. In the adjoint representation this background is given by $F_{mn} = \mathrm{diag}(0, ..., 0, \mathrm{F}_{mn}, -\mathrm{F}_{mn}, ..., \mathrm{F}_{mn}, -\mathrm{F}_{mn})$ with the total

number of non-zero entries $2\mathbf{N} = 2n_0 N_0$ and $\mathrm{F}_{mn} = \tilde{\mathcal{F}}^{(1)}_{mn} - \tilde{\mathcal{F}}^{(2)}_{mn}$ (which has block-diagonal structure as the non-zero components of $\tilde{\mathcal{F}}^{(1)}_{mn}$ and $\tilde{\mathcal{F}}^{(2)}_{mn}$ are orthogonal). As a result, here $l = 2,\ \tilde{f}_1 = \tilde{f},\ \tilde{f}_2 = iv$ and thus $\delta = -iW \sim \frac{1}{v\tilde{f}r^4}(\tilde{f}^4 + 2v^2\tilde{f}^2 + v^4)$ in agreement with [11].

4.2 Interaction of 'D-string' with 3-brane–instanton bound state

The matrix model background corresponding to the configuration of the 'D-string' $(1+i)$ wrapped over a 2-torus in $(5,6)$ directions and the D3-brane–D-instanton bound state $(3\|i)$ wrapped over a 4-torus in $(1,2,3,4)$ directions, which are separated by a distance b in the 9-direction is given by $(a = 1, ..., 4;\ T = 1)$

$$\bar{X}^{(1)}_a = T^{-1}(i\partial_a + \tilde{A}_a) = P_a\,, \qquad \bar{X}^{(1)}_9 = b\,, \tag{4.18}$$

$$\bar{X}^{(2)}_5 = q\,, \qquad \bar{X}^{(2)}_6 = p\,, \qquad [q,p] = i\tilde{f}\mathrm{I}\ ,$$

where the $U(N_{-1})$ gauge potential $\tilde{A}_a$ is representing the charge N_3 instanton on the dual torus $\tilde{T}^4$ (see, e.g., [9]), i.e. its field strength $G_{ab} = \partial_a\tilde{A}_b - \partial_b\tilde{A}_a - i[\tilde{A}_a, \tilde{A}_b]$ satisfies

$$G_{ab} = *G_{ab}\ , \qquad \frac{1}{16\pi^2}\int_{\tilde{T}^4} d^4x\ \mathrm{tr}(G_{ab}G_{ab}) = N_3\,. \tag{4.19}$$

The bosonic, ghost and fermionic contributions to the effective action (4.5) in this background are

$$W_B = \sum_{n=0}^{\infty} \mathrm{Tr}\ \ln\Big[(b_n^2 + P^2)\delta_{\mu\nu} - 2i\mathcal{F}'_{\mu\nu}\Big]\ , \qquad W_G = -2\sum_{n=0}^{\infty} \mathrm{Tr}\ \ln\Big(b_n^2 + P^2\Big)\ , \tag{4.20}$$

$$W_F = -\tfrac{1}{2}\sum_{n=0}^{\infty} \mathrm{Tr}\ \ln\Big(b_n^2 + P^2 + \frac{i}{2}\gamma_{mk}\mathcal{F}'_{mk}\Big)\,, \qquad b_n^2 \equiv b^2 + \tilde{f}(2n+1)\ .$$

where

$$\mathcal{F}'_{mn} = \begin{pmatrix} G_{ab} & 0 \\ 0 & \tilde{\mathcal{F}}_{\alpha\beta} \end{pmatrix}, \qquad \tilde{\mathcal{F}}_{\alpha\beta} = \tilde{f}\epsilon_{\alpha\beta}\ . \tag{4.21}$$

The expression for W is computed in a similar way as in the T-dual case of (2+0)–(4‖0) configuration considered in [18]. The final result for the leading long-distance $(b\to\infty)$ term in W is

$$W = \frac{1}{32\pi^2 b^2}\Big[\tilde{f}\int_{\tilde{T}^4} d^4x\ \mathrm{tr}(G_{ab}G_{ab}) - \tilde{V}_4 N_{-1}\tilde{f}^3\Big] + O(\frac{1}{b^4}) \tag{4.22}$$

$$= \frac{1}{2b^2}\Big(N_3\tilde{f} - \tfrac{1}{16\pi^2}\tilde{V}_4 N_{-1}\tilde{f}^3\Big) + O(\frac{1}{b^4})\ .$$

This becomes exactly the same as the supergravity result for the interaction potential (2.21) after we set $b = r,\ \tilde{f} = f^{-1},\ n_1 = 1$, use the relation (2.17), i.e. $\tilde{V}_4 V_4 = (2\pi)^4$, and note that since it is assumed that $N_{-1} \gg N_3$ the last term in (2.21) can be neglected.

The expression (4.22) is equivalent to the leading-order $O(F^4)$ term in the $U(n_{-1}+N_{-1})$ SYM effective action (3.9) on the dual 6-torus $\tilde{T}^2 \times \tilde{T}^4$ computed in the background

$$F_{mn} = \hat{\mathcal{F}}_{mn} = \begin{pmatrix} G_{ab} & 0 \\ 0 & \tilde{\mathcal{F}}_{\alpha\beta} \mathrm{I}_{n_{-1}\times n_{-1}} \end{pmatrix} . \quad (4.23)$$

Indeed, substituting $\hat{\mathcal{F}}_{mn}$ (4.23) into (3.10), i.e. into $\mathbf{b}_8$ in (3.5), and observing that the G^4-terms cancel out ($\mathbf{b}_8$ vanishes on a self-dual gauge field background) one is left with the the abelian $\tilde{\mathcal{F}}^4$ term and the 'cross-term' $\tilde{\mathcal{F}}^2 G^2$, i.e.

$$\mathbf{b}_8(\hat{\mathcal{F}}) = \mathrm{Tr}\Big[\tilde{\mathcal{F}}^4 - \tfrac{1}{4}(\tilde{\mathcal{F}}^2)^2 - \tfrac{1}{2}\tilde{\mathcal{F}}^2 G^2\Big] = 2n_{-1}\Big[N_{-1}\tilde{f}^4 - \tilde{f}^2 \mathrm{tr}(G_{ab}G_{ab})\Big] , \quad (4.24)$$

where in the first expression $\tilde{\mathcal{F}}$ and G are the adjoint representation counterparts of the $N \times N$ matrices in the fundamental representation with non-vanishing $n_{-1} \times n_{-1}$ and $N_{-1} \times N_{-1}$ blocks (note that the spatial components of $\tilde{\mathcal{F}}$ and G are orthogonal). One can also derive this expression by formally representing G_{ab} as an abelian matrix with two equal 2×2 blocks, i.e. $G_{ab} = g\epsilon_{ab}\mathrm{I}_{N_{-1}\times N_{-1}}$ for $a, b = 1, 2$ and $a, b = 3, 4$. Then one may apply formula (3.20) for the abelian background with $\mathrm{f}_1 = \tilde{f}$, $\mathrm{f}_2 = \mathrm{f}_3 = g$. This gives $\mathbf{b}_8 = 2n_{-1}N_{-1}[2(\tilde{f}^4 + 2g^4) - (\tilde{f}^2 + 2g^2)^2] = 2n_{-1}N_{-1}(\tilde{f}^4 - 4\tilde{f}^2 g^2)$, i.e. the same result as in (4.24) since $\mathrm{tr}(G_{ab}G_{ab}) = 4N_{-1}g^2$, $(2\pi)^{-2}\tilde{V}_4 g^2 N_{-1} = N_3$ (cf. (2.4)).

For $n_1 = 1$ one has $n_{-1} = 2\pi\tilde{V}_2^{-1}\tilde{f}^{-1}$ (see (2.22)) and concludes that

$$\Gamma_M = -\frac{1}{2(4\pi)^3 M^2}\tilde{V}_2 \int_{\tilde{T}^4} d^4x \ \mathbf{b}_8 + O(\frac{1}{M^4}) \quad (4.25)$$

in (3.9) is equal to W in (4.22) for $b = M$. This is also in agreement with the supergravity potential represented in the form (2.23).

Thus we have found complete agreement between the 1-loop matrix model and classical supergravity expressions for the leading-order long-distance interaction potentials.

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